TAUBERIAN THEOREMS AND TAUBERIAN CONDITIONS

BY
G. G. LORENTZ

1. **Introduction.** The chief aim of this paper is an investigation of relations among Tauberian theorems. In §2 we compare " high indices theorems" or "gap theorems" with "order" Tauberian theorems containing a condition of the form $u_n = o(c_n)$ or $u_n = O(c_n)$. Especially for the methods of Abel and Cesàro we shall look for necessary and sufficient conditions on the numbers c_n, for which these theorems are valid.

2. **Relations among Tauberian theorems.** Let A be a Toeplitz-Silverman method of summation, given by the transformation

$$\sigma_m = a_{m1}s_1 + a_{m2}s_2 + \cdots + a_{mn}s_n + \cdots \tag{1}$$

of the sequence s_n into the sequence σ_m. We shall call the sequence s_n and the series $\sum u_n$ with the partial sums s_m A-summable to the value s, if $\lim \sigma_m = s$. We assume that the conditions of regularity for the method A are fulfilled. Then from $s_n \to s$ we have $\sigma_m \to s$. A *Tauberian theorem* for the method A is a proposition in which, conversely, $s_n \to s$ is deduced from $\sigma_m \to s$ and an additional condition on the series $\sum u_n$ or the sequences s_n. This latter condition is called a *Tauberian condition* for the method A.

Let n_k be a sequence $n_1 < n_2 < \cdots$ of positive integers. The following propositions are called high indices or gap theorems for the method A.

(H$_1$). *If a series $\sum u_n$ is A-summable and satisfies the gap condition*

$$u_n = 0, \qquad n \neq n_1, n_2, \cdots, \tag{2}$$

then the series is convergent.

(H$_2$). *If the A-transform σ_m of the sequence $s_n = \sum_{k=1}^{n} u_k$ is bounded and if the series $\sum u_n$ satisfies* (2), *then s_n is also bounded.*

Let c_n be any sequence of numbers $0 \leqq c_n \leqq +\infty$. In analogy to the above propositions, we state the following Tauberian theorems containing an estimate of u_n.

(T$_1$). *If a series $\sum u_n$ is A-summable and satisfies the order condition*

$$u_n = o(c_n) \tag{3}$$

then the series is convergent.

(T$_2$). *If σ_n is bounded and*

$$u_n = O(c_n) \tag{4}$$

then s_n is also bounded.

Presented to the Society, October 25, 1947; received by the editors February 18, 1947.

Note that (3) and (4) impose no limitation for those u_n for which $c_n=+\infty$. Hence it may happen that a high indices theorem is contained in an order Tauberian theorem. See Theorem 3 which, for the Abel method, contains as special cases the Tauberian theorem with the condition $u_n=O(1/n)$ and the Hardy-Littlewood high indices theorem.

We now prove the following theorem.

THEOREM 1. *If for the method A the high indices theorem* (H_1) *is valid, and if further*

$$\sum_{n_k<n<n_{k+1}} c_n = O(1), \tag{5}$$

then (T_1) *is also valid.*

Proof. Suppose that, for an A-summable series $\sum u_n$, the condition (3) is fulfilled. Let $s_n'=0$ when $n<n_1$ and $s_n'=s_{n_k}$ when $n_k\leqq n<n_{k+1}$, $k=1, 2, \cdots$. Then by means of the estimates (3) and (5) we get $s_n-s_n'\to 0$. From the regularity of the method A we now have for the A-transform σ_m' of the sequence s_n'

$$\sigma_m - \sigma_m' = \sum_{n=1}^{\infty} a_{mn}(s_n - s_n') \to 0. \tag{6}$$

Thus lim σ_m' exists, as according to the supposition lim σ_m exists.

The convergence of s_n' then follows from (H_1), and because of $s_n-s_n'\to 0$ the sequence s_n is also convergent.

THEOREM 1*. *If for the method A the high indices theorem* (H_1) *is valid for all series satisfying the additional condition*

$$s_n = o(k(n)),$$

$k=k(n)$ *being defined by* $n_k\leqq n<n_{k+1}$ *and if*

$$\sum_{n_k\leqq n<n_{k+1}} c_n = O(1), \tag{7}$$

then (T_1) *also is valid for A.*

The proof is given in the same manner as above, the only difference being that we have to prove that $s_n'=o(k(n))$ for the sequence s_n', in order to be able to use the theorem (H_1). But according to (3) and (5), $s_n=o(k(n))$, hence the same estimate holds also for s_n'.

Similarly we can prove the following theorem.

THEOREM 2. *The proposition* (T_2) *is a consequence of* (H_2) *if the* c_n *satisfy the condition* (5).

THEOREM 2*. *The proposition* (T_2) *is a consequence of* (H_2) *for all series* *satisfying* (7) *and the additional condition*

$$s_n = O(k(n)).$$

A further relation among Tauberian theorems can be obtained by means of the following theorem of Mazur and Orlicz [9] (1); we do not possess any proof of this theorem. *If, for a regular method A, no unbounded A-summable sequences exist, then only convergent sequences are A-summable.*

By application of this theorem to the method $\tilde{A}$ with matrix

$$\tilde{a}_{mk} = \sum_{n_k \leqq n < n_{k+1}} a_{mn}$$

we obtain the following theorem. *The high indices theorem* (H_1) *follows from* (H_2).

In spite of their simplicity, these theorems seem to offer a certain amount of interest. By means of Theorem 1, o-Tauberian theorems can be derived with exact order of the u_n. Thus for example we deduce from the high indices theorem of Hardy and Littlewood [3], according to which (H_1) is true for the Abel (or Euler) power series method P when $n_k = 2^k$, the following theorem of Tauber. If the series $\sum u_n$ is P-summable and $u_n = o(1/n)$, then the series is convergent.

Furthermore, by use of a theorem of Pitt [12] and Theorem 1* we see that the estimate $u_n = o(n^{-1/2})$ implies the convergence of $\sum u_n$ when $\sum u_n$ is Borel summable. Pitt's theorem states that if, for a B-summable series $\sum u_n$, (2) is valid with $n_{k+1} - n_k \geqq a(n_k)^{-1/2}$, $a > 0$, and $s_n = O(\lambda^n)$ for every $\lambda > 1$, then $\sum u_n$ is convergent. In fact we have only to observe that from $n_{k+1} - n_k \geqq a(n_k)^{1/2}$, $a > 0$, we can easily derive $n_k \geqq k^2/q^2$ with a certain constant $q > 0$. This means that for the function $k(n)$, defined in Theorem 1*, we have $k(n) \leqq qn^{1/2}$. For a result, similar to Pitt's, for the method E_1 of Euler-Knopp, see Meyer-König [11].

It is scarcely possible to derive, by the same simple method, the exact O-Tauberian theorems containing the conditions $u_n = O(1/n)$ and $O(1/n^{1/2})$ respectively. These O-Tauberian theorems are connected with much more delicate properties of the matrix of a transformation than those made use of in Theorem 1. See, for example, Karamata [6, p. 20].

We shall deduce from Theorem 1* that Pitt's theorem cannot in its essentials be rendered more precise. For each $\epsilon > 0$, $a > 0$, and an increasing sequence of indices n_k with

$$n_{k+1} - n_k \leqq a n_k^{1/2-\epsilon} \qquad (k = 1, 2, \cdots)$$

a divergent B-summable series exists, satisfying $s_n = o(n)$ and (2). For obviously $k(n) \leqq n$. If there were no such series, then Theorem (H_1) with the

(1) Numbers in brackets refer to the references cited at the end of the paper.

additional condition $s_n=o(k(n))$ would be correct for the method B. Let $c_n=n^{-1/2+\epsilon}$, then (7) is fulfilled:

$$\sum_{n_k\leqq n<n_{k+1}} c_n \leqq n_k^{-1/2+\epsilon}(n_{k+1}-n_k+1)=O(1). \tag{8}$$

Then according to Theorem 1* the condition $u_n=O(n^{-1/2+\epsilon})$ would be a Tauberian condition, and this for the method B is not the case for any $\epsilon>0$ according to Hardy and Littlewood [2, p. 15].

3. The general forms of the O-Tauberian theorem for the Abel and Cesàro methods. We shall say that a sequence c_n, $0\leqq c_n\leqq+\infty$ has the property (E) when

(E) For each $\epsilon>0$ there is a sequence of positive numbers n_k for which $n_{k+1}/n_k\geqq q>1$ and

$$\sum_{n_k<n<n_{k+1}} c_n<\epsilon.$$

It obviously does not matter whether we suppose in this condition the n_k to be any positive number or only integers. The supposition chosen renders the proof slightly easier. For treatments of general Tauberian conditions, see Pitt [12] and Agnew [1].

THEOREM 3. *The condition* (E) *is the necessary and sufficient condition for the sequence* $0\leqq c_n\leqq+\infty$ *in order that*

$$u_n=O(c_n) \tag{9}$$

is a Tauberian condition for the methods C_α $(\alpha>0)$ *of Cesàro or the Abel method* P.

Proof. (a) *The condition* (E) *is necessary.* We investigate the method C_1 and shall show first of all that, $\epsilon>0$ being given, the indices n_k for which $c_{n_k}\geqq\epsilon$ constitute a sequence with $n_{k+1}/n_k\geqq q>1$, if there is an infinite number of them. Otherwise we should have an $\epsilon_0>0$ and two sequences of integers m_k, l_k with $m_k<l_k<m_{k+1}$, $l_k/m_k\to1$, $c_{m_k}\geqq\epsilon_0$, $c_{l_k}\geqq\epsilon_0$. By considering a partial sequence we can regard the following as fulfilled:

$$[(l_1-m_1)+(l_2-m_2)+\cdots+(l_k-m_k)]/m_k\to0.$$

Under these conditions, let

$$\begin{aligned} u_n &= +\epsilon_0 && (n=m_k,\ k=1,2,\cdots)\\ &= -\epsilon_0 && (n=l_k,\ k=1,2,\cdots)\\ &= 0 && (\text{for all remainlng } n). \end{aligned}$$

Then for the C_1 transform σ_n of the series $\sum u_n$ we have, when $m_k\leqq n<m_{k+1}$,

$$0\leqq\sigma_n\leqq[(l_1-m_1)+\cdots+(l_k-m_k)]\epsilon_0/m_k\to0.$$

Furthermore $|u_n| \leqq c_n$. As the series $\sum u_n$ diverges, (9) would not be a Tauberian condition for the method C_1.

We shall now prove, again for the method C_1, that the sequence c_n has the property (E). Suppose (E) is not fulfilled. Then an $\epsilon_0 > 0$ exists for which there is no sequence of positive numbers n_k which fulfills the requirements of (E). Thus for every sequence n_k, with $n_{k+1}/n_k \geqq q > 1$, $\sum_{n_k < n < n_{k+1}} c_n \geqq \epsilon_0$ holds even for an infinity of k.

Let n_k^0 be a sequence as mentioned above, for which $c_n < \epsilon_0/3$ for $n \neq n_k^0$. We can suppose

$$1 < q_0 \leqq \overset{0}{n}_{k+1}/\overset{0}{n}_k \leqq Q_0 < +\infty \qquad (k = 1, 2, \cdots).$$

From the sequence n_k^0 we form the sequence n_k^1 whose elements are

$$\overset{0}{n}_1,\ (\overset{0}{n}_1\overset{0}{n}_2)^{1/2},\ \overset{0}{n}_2,\ (\overset{0}{n}_2\overset{0}{n}_3)^{1/2},\ \overset{0}{n}_3, \cdots .$$

Similarly from n_k^1 we form the sequence n_k^2, and so on. Then for each sequence n_k^p we evidently have

$$1 < q_p \leqq \overset{p}{n}_{k+1}/\overset{p}{n}_k \leqq Q < \infty \qquad (k = 1, 2, \cdots)$$

where

$$q_p = (q_0)^{1/2^p}, \qquad Q_p = (Q_0)^{1/2^p}. \tag{10}$$

According to the above, we can find numbers

$$N_s = \overset{p_s}{n}_{k_s}, \qquad N_s' = \overset{p_s}{n}_{k_s+1}, \qquad (s = 1, 2, \cdots)$$

such that

$$N_s \to \infty, \qquad p_s \to \infty, \qquad N_s < N_s' < N_{s+1}, \qquad \sum_{N_s < n < N_s'} c_n \geqq \epsilon_0. \tag{11}$$

Since, according to (10), as $s \to \infty$,

$$0 \leqq \frac{1}{N_s}(N_s' - N_s) = \frac{N_s'}{N_s} - 1 \leqq Q_{p_s} - 1 \to 0$$

we can even achieve

$$[(N_1' - N_1) + (N_2' - N_2) + \cdots + (N_s' - N_s)]/N_s \to 0$$

by taking a partial sequence of the s.

As $c_n < \epsilon_{0/3}$ for $N_s < n < N_s'$ and on account of the last inequality (11) there is an $\overline{N}_s$ between N_s and N_s' such that

$$\sum_{N_s < n \leqq \overline{N}_s} c_n \geqq \epsilon_0/3, \qquad \sum_{\overline{N}_s < n < N_s'} c_n \geqq \epsilon_0/3.$$

(One has merely to choose a first $\overline{N}_s$ for which the first of these inequalities is fulfilled.)

We now define u_n to be positive in $N_s < n \leqq \overline{N}_s$ and negative in $\overline{N}_s < n < N_s'$ such that $|u_n| \leqq c_n$ and

$$\sum_{N_s < n \leqq \overline{N}_s} u_n = \epsilon_0/3, \qquad \sum_{\overline{N}_s < n < N_s'} c_n = -\epsilon_0/3.$$

For the remaining n let $u_n = 0$. Then $\sum u_n$ is divergent and $|u_n| \leqq c_n$ holds for all n, but σ_m converges toward zero. For we have $s_n = 0$ for any n outside the intervals $N_s < n < N_s'$ and $0 \leqq s_n \leqq \epsilon_0/3$ within these intervals. Thus for $N_s \leqq n < N_{s+1}$ we have

$$0 \leqq \sigma_n \leqq [(N_1' - N_1) + (N_2' - N_2) + \cdots + (N_s' - N_s)]\epsilon_0/3N_s \to 0.$$

Hence (9) is not a Tauberian condition for the method C_1. We thus have a contradiction.

Regarding the methods C_α $(\alpha > 0)$ and P, the necessity of (E) follows for them from the fact that they contain the method C_1. Finally according to a theorem by Andersen the methods C_α $(\alpha > 0)$ and C_1 are equivalent for series with bounded partial sums; see, for example, Zygmund [14, p. 262]. Hence the above proof, making use of series of this kind only, remains valid for C_α, $0 < \alpha < 1$, as well as for C_1.

(b) *We shall now show that* (E) *is sufficient.* This part of Theorem 3 is not new. It follows from a theorem of Pitt [12, Theorem 13]. Pitt considers more general methods, and his proof is much more complicated than the one given here. A proof of Agnew [1, Theorem 9.21] of a C_1 Tauberian theorem, with a very general Tauberian condition, is more like the proof given below. Suppose (E) and (9) to be fulfilled, and let s_n be P-summable. Since Ingham [4] has shown that the high indices theorem (H_2) for the Abel method is valid for any sequence $\{n_k\}$ with $n_{k+1}/n_k \geqq q > 1$, we see in accordance with Theorem 2 that the sequence s_n is bounded. As the methods P and C_1 are equivalent for such sequences (see, for example, Landau [7, p. 12]), we first obtain the C_1-summability of the sequence s_n. Its convergence follows in the known manner:

Without restriction of generality, we can suppose $|u_n| \leqq c_n$ and $\sigma_n = (s_1 + \cdots + s_n)/n \to 0$. We choose any $\epsilon > 0$ and a sequence n_k in accordance with (E). For all k_0 sufficiently large we have $|\sigma_n| < \epsilon$ for $n \geqq n_{k_0} - 1$. Hence from

$$\sigma_{n_{k+1}-1} = \frac{s_1 + \cdots + s_{n_k - 1}}{n_{k+1} - 1} + \frac{s_{n_k} + \cdots + s_{n_{k+1}-1}}{n_{k+1} - 1}$$

for $k \geqq k_0$, $n_k \leqq n < n_{k+1}$ we have, on account of $|s_m - s_n| < \epsilon$ for $n_k \leqq m < n_{k+1}$, that

$$\sigma_{n_{k+1}-1} = \frac{n_k - 1}{n_{k+1} - 1}\sigma_{n_k - 1} + \frac{n_{k+1} - n_k}{n_{k+1} - 1}s_n + \theta\frac{n_{k+1} - n_k}{n_{k+1} - 1}\epsilon$$

where $|\theta|<1$. Thence we deduce for the s_n

$$|s_n| \leqq \epsilon + \frac{2n_{k+1}}{n_{k+1}-n_k}\epsilon \leqq \left(1+\frac{2q}{q-1}\right)\epsilon,$$

that is, $s_n \to 0$. This completes the proof of Theorem 3.

An analogous theorem can be proved for the one-sided Tauberian condition. Instead of condition (E), we now introduce condition (F) for the sequence $0 \leqq c_n \leqq +\infty$ which shall signify:

(F) For each $\epsilon>0$, there is a sequence n_k of positive numbers with $n_{k+1}/n_k \geqq q > 1$ for which

$$\sum_{n_k \leqq n < n_{k+1}} c_n < \epsilon.$$

Thus while in (E) the possibility $c_n = +\infty$ for an infinity of n was not excluded, nearly all c_n are finite in this case. The theorem mentioned can be stated as follows:

THEOREM 4. *The condition* (F) *is the necessary and sufficient condition for the sequence* $0 \leqq c_n \leqq +\infty$ *in order that*

$$u_n \leqq O(c_n) \tag{12}$$

is a Tauberian conditions for the methods C_α $(\alpha>0)$ *of Cesàro or the Abel method* P.

Proof. (a) *The condition* (F) *is necessary.* We shall first prove that $c_n \to 0$. Otherwise there would be a sequence of indices $l_k \to \infty$ with $c_{l_k} \geqq \epsilon_0 > 0$. We may suppose $l_{k+1} > l_k + 1$ and furthermore

$$k/l_k \to 0 \qquad \text{for } k \to \infty. \tag{13}$$

Let $u_n = -1$ when $n = l_k - 1$, $k = 2, 3, \cdots$; $u_n = +1$ when $n = l_k$, $k = 2, 3, \cdots$; and $u_n = 0$ for all remaining n. We have $u_n \leqq O(c_n)$ and the respective s_n constitute a bounded sequence. This sequence is C_1-summable on account of (13), hence also C_α $(\alpha>0)$ and A-summable. But it is divergent and therefore (12) is not a Tauberian condition.

Now according to Theorem 3 the condition (E) must be fulfilled. This together with $c_n \to 0$ gives (F), if we omit some of the first n_k if necessary.

(b) *The sufficiency of* (F) can be deduced from a theorem by R. Schmidt [13, Theorem 11] according to which

$$\limsup_{\delta \to 0} \phi(\delta) \leqq 0; \qquad \phi(\delta) = \limsup_{m \to \infty} \max_{m \leqq v \leqq w \leqq m(1+\delta)} (s_w - s_v) \tag{14}$$

represents a Tauberian condition for the Abel method.

Suppose (F) to be fulfilled. We then choose the sequence n_k for a given $\epsilon>0$ in accordance with (F). If then $\delta>0$ is so small that $1+\delta<q$ and if

$n_k \leqq m < n_{k+1}$, then surely $m(1+\delta) < n_{k+2}$ and therefore

$$s_w - s_v = u_{v+1} + \cdots + u_w \leqq M \sum_{n_k \leqq n < n_{k+2}} c_n < 2M\epsilon$$

with a constant $M>0$ for $m \leqq v \leqq w \leqq m(1+\delta)$. Thus $\phi(\delta) \leqq 2M\epsilon$ and (14) is fulfilled. This completes the proof of Theorem 4.

Now we shall add some consequences to illustrate the applicability of the theorems proved above.

1. Let $\omega(n) \to \infty$ for $n \to \infty$. Then $u_n = O(\omega(n)/n)$ is not a Tauberian condition for the methods C_α $(\alpha > 0)$ and P. (For the Abel method, this was proved by Littlewood [8].) For the condition (E) is not fulfilled here:

$$\sum_{n_k < n < n_{k+1}} c_n = \sum_{n_k < n < n_{k+1}} \omega(n)/n \geqq \omega(n_k)(n_{k+1} - n_k - 2)/n_{k+1}$$

$$\geqq \omega(n_k)\left(1 - q - \frac{2}{n_{k+1}}\right) \to \infty.$$

2. If c_n is not increasing, $u_n = O(c_n)$ or also $u_n \leqq O(c_n)$ is then and then only a Tauberian condition, when for every $\epsilon > 0$ there is a $\delta > 0$ such that for all n sufficiently large

$$\sum_{n < v < m(1+\delta)} c_v < \epsilon. \tag{15}$$

For (F) follows from this property of the c_n if we choose $n_k = (1+\delta)^k$ and add sufficiently many of the first integers to these n_k. Conversely, (15) follows from (F) or (E) as can easily be proved on the same lines as (b), Theorem 4.

3. Menchoff [10] published a theorem which implies that $u_n = O(c_n)$ with

$$\sum_{n=2^k}^{n=2^{k+1}} c_n = O(1) \tag{16}$$

is a Tauberian condition for the method P. This theorem was later withdrawn by him [10]. Now it is easy to form a sequence c_n for which (16) holds although (E) is not fulfilled. (For instance, let $c_{2^k} = c_{2^k+1} = 1$, and $c_n = 0$ for other n.) Hence (16) is indeed not a Tauberian condition.

4. If n_k is a sequence with $n_{k+1}/n_k \geqq q > 1$ and if for a series $\sum u_n$ which is P-summable, $u_n = O(1/n)$ holds for $n \neq n_1, n_2, \cdots$, then the series is convergent. (For the Cesàro methods C_α, see Meyer-König [11].)

Let $c_n = M/n$ for $n \neq n_k$, $c_n = +\infty$ for $n = n_1, n_2, \cdots$. For a given $\epsilon > 0$ we then choose the sequence m_k such that it contains all the n_k and that $1 < q' \leqq m_{k+1}/m_k < 1 + \epsilon$. Then for every k we have

$$\sum_{m_k < n < m_{k+1}} c_n \leqq \frac{M}{m_k}(m_{k+1} - m_k) < M\epsilon,$$

that is, the condition (E) is fulfilled.

5. Suppose $M_n \geqq 0$ to be a C_1-summable sequence. Then

$$u_n \leqq M_n/n$$

is a Tauberian condition for the methods C_α and P. (See Karamata [5].)

We have only to prove that the numbers $c_n = M_n/n$ satisfy the condition (F). For a given $\epsilon > 0$ let $n_k = (1+\epsilon)^k$. We shall designate $\sigma_k = (1/n_k) \sum_{n<n_k} M_n$ and $\sigma = \lim \sigma_k$. Then

$$\sum_{n_k \leqq n < n_{k+}} c_n \leqq \frac{1}{n_k} \sum_{n_k \leqq n < n_{k+1}} M_n = \frac{1}{n_k}(n_{k+1}\sigma_{k+1} - n_k\sigma_k)$$
$$= \epsilon\sigma_{k+1} + (\sigma_{k+1} - \sigma_k).$$

For k sufficiently large $|\sigma_{k+1} - \sigma_k| < \epsilon$, $\sigma_{k+1} < \sigma + 1$ and therefore the sum to be estimated is less than

$$\epsilon(\sigma + 1) + \epsilon = \epsilon(\sigma + 2).$$

References

1. R. P. Agnew, *Tauberian conditions*, Ann. of Math. vol. 42 (1941) pp. 293–308.

2. G. H. Hardy and J. E. Littlewood, *The relations between Borel's and Cesàro's methods of summation*, Proc. London Math. Soc. vol. 11 (1913) pp. 1–16.

3. ———, *A further note on the converse of Abel's theorem*, Proc. London Math. Soc. vol. 25 (1926) pp. 219–236.

4. A. E. Ingham, *On the "high-indices theorem" of Hardy and Littlewood*. Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 1–7.

5. J. Karamata, *Théorèmes inverses de sommabilité* I, II, Glas Srpske Kralevske Akad., Beograd, vol. 143 (70) (1931) pp. 3–24, 121–146.

6. ———, *Sur les théorèmes inverses des procédés de sommabilité*, Paris, 1937, 46 pp.

7. E. Landau, *Neuere Ergebnisse der Funktionentheorie*, 2d ed., Berlin, 1929, 122 pp.

8. J. E. Littlewood, *The converse of Abel's theorem on power series*, Proc. London Math. Soc. vol. 9 (1911) pp. 434–443.

9. S. Mazur and W. Orlicz, *Sur les méthodes linéaires de sommation*, C. R. Acad. Sci. Paris vol. 196 (1933) pp. 32–34.

10. D. Menchoff, *Sur une généralisation d'une théorème de M. M. Hardy et Littlewood*, Rec. Math. (Mat. Sbornik) N.S. vol. 3 (1938) pp. 367–373; vol. 5 (1939) p. 451.

11. W. Meyer-König, *Limitierungsumkehrsätze mit Lückenbedingungen* I; II, Math. Zeit. vol. 45 (1939) pp. 447–478; 479–494.

12. H. R. Pitt, *General Tauberian theorems*, Proc. London Math. Soc. vol. 44 (1938) pp. 243–288.

13. R. Schmidt, *Über divergenten Folgen und lineare Mittelbildungen*, Math. Zeit. vol. 22 (1925) pp. 89–152.

14. A. Zygmund, *Trigonometrical series*, Warsaw-Lwów, 1935 331 pp.

FRANKFURT, GERMANY.